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Jean-David Benamou

N° 2688
Octobre 1995

PROGRAMME 6

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Big Ray Tracing : Multivalued Travel Time Field Computation using Viscosity Solutions of the Eikonal Equation

Jean-David Benamou

Programme 6 — Calcul scientifique, modélisation et logiciel numérique
Projet Ident

Rapport de recherche n° 2688 — Octobre 1995 — 31 pages

Abstract: An hybrid method based on a finite difference upwind scheme and ray tracing is presented. It allows an easy and fast computation of multivalued travel time fields for high frequency signals.
keywords : Hamilton-Jacobi, Calculus of Variation, Fermat Principle, Geometrical Optics, Ray Tracing, Multivalued Travel Time Field, Viscosity Solution, FD/FE Upwind Scheme, High Frequency Asymptotic, Wave equation, Eikonal equation, Hyperbolic Equation.

AMS(MOS) subject classification : 34H05, 49S05, 65N06, 65D99, 73D99, 78-08, 78A05.

(Résumé : tsvp)

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : (33 1) 39 63 55 11 – Télécopie : (33 1) 39 63 53 30

Gros Lancer de Rayons : Calcul de Temps d'Arrivées Multiples à l'Aide de Solutions de Viscosité de l'Equation Eikonale

Résumé : Une méthode hybride basée sur un schéma décentré aux différences finies et le lancer de rayons est présentée. Elle permet le calcul facile et rapide des temps d'arrivées multiples de signaux à hautes fréquences.
Mots-Clés : Hamilton-Jacobi, Calcul des Variations, Principe de Fermat, Optique Géometrique, Lancer de Rayons, Temps d'Arrivées Multiples, Solution de Viscosité, FD/FE schéma Décentré, Asymptotiques Haute Fréquence, Equation des Ondes, Equation Eikonale, Equation Hyperbolique

1 Introduction

Travel (or arrival) times of electromagnetic or acoustic high frequency signals are classically computed using ray tracing [10] [29] [7]. Rays are shot from a source point in all directions and are paths, solutions of ordinary differential equations. Travel times at each point are given by integrating the slowness index locally characterizing the medium along these paths. For complicated velocity (inverse of slowness) profiles, rays may cross and multiple travel times occur at the same location. These ray solutions have to be interpolated in order to recover multivalued travel time field in the whole domain. Such methods have been proposed for example in [29] [11] [13] [27] [16].

Rays can also be interpreted as the characteristics of a Hamilton-Jacobi (hyperbolic) equation called the Eikonal equation. In the case of a multivalued travel time field the Eikonal equation has a matching multivalued solution. These rays have a relevant physical meaning in a wide class of velocity profiles (see [8] for a discussion on this topic).

Because of potential applications in geophysical inverse problems [22] [2], interest has arisen for a finite difference resolution of the Eikonal equation [28] [21] [15] [20] [30]... Though easier to implement and faster than ray tracing, upwind finite difference / finite elements schemes only compute a single-valued solution also known as the viscosity solution which corresponds in the case of multivaluedness to the earliest travel time (or first arrival time).

The computation of latest travel times is nevertheless important from the application point of view. The computation of multivalued travel time fields is also a challenging academic problem which has already received some attention [31] [1] [4] [21].

We propose an original approach based on a theorem of P.-L. Lions [18] which characterizes the viscosity solution of the Eikonal equation at each point on a given domain as the minimum length (length being here the travel time) over all the paths contained in the considered domain from the source to the considered point.

This minimization problem is of course equivalent to the Fermat principle. The Euler equations characterizing the extrema of this variational problem are the ray equations. A straightforward application of this theorem to the whole domain of study characterizes the viscosity solution as the minimum length with respect to all possible paths and hence the earliest travel time.

If multivaluedness occurs at one point along a ray, and if this ray is a local minimum corresponding to a latest arrival time, it is also a global minimum with respect to the path restricted to lie in a local domain around this ray. Applying again Lions's theorem, we see that each of these latest arrival times is the viscosity solution of the Eikonal equation set on this restricted local domain.

Hidden behind this theory is the simple physical idea that the viscosity solution follows the wave generated by by a given source. When more waves are generated by the inhomogeneities of the velocity (reflected, refracted ...) and these waves interact, shocks appear in the viscosity solution and only the first arrival is picked up. Selecting our restricted local domain around a ray, we get rid of possible interferences caused by outside waves entering this domain and the viscosity solution gives the travel time associated with the local wave which is a latest arrival time.

There is still a gap between these theoretical results and a tractable numerical algorithm which can be applied without a priori knowledge on the number and location of the different travel times.

Our proposed algorithm consists in :

- Tracing a given number of rays shot in regularly spaced initial directions.
- Using a finite difference solver to compute the viscosity solution on a regular grid of the Eikonal equation in restricted domains obtained by generating an envelope around two successively shot rays. We call these envelopes 'big rays'.
- The travel time field is given by the superposition of all local travel time fields computed in the big rays. When big rays intersect (i.e. when rays are crossing), this process gives a multivalued travel time field.

At fixed number of rays, the ability of this algorithm to recover the different travel times clearly depends on the velocity profile. Increasing the number of rays we simply tend to plain ray tracing. The accuracy of the computed travel times depends on the fineness of the finite difference grid.

The paper is organized as follows : Section 2 reviews the different formulations of the travel time problem going from the Fermat principle to the Eikonal equation via the Hamiltonian formalism. Section 3 presents Lions's theorem and discusses its application to local and global minima of the shortest travel time problem. We present in section 4 sufficient conditions for the rays to be local minima. Section 3 and 4 constitute a partial theoretical justification of our method. We propose an algorithm in section 5 and discuss its validity. Section 6 presents experiments done a simple test case with a layered velocity profile.

2 From the travel time problem to the Hamilton-Jacobi equation

The oldest and classical definition of the travel time of an optic signal going through a medium with a given velocity (or slowness, its inverse) is the Fermat principle. It states that the travel time for the signal to go from a point A to a point B is given as the infimum of the functional

$$J[y] = \int_A^B n(y(s))\, ds$$

over all the paths y going from A to B. $n = \frac{1}{c}$ is the slowness (c is the velocity) and s is the curvilinear abscissa. As c is taken strictly positive this infimum always exists. We assume throughout this paper that n is continuous up to its second derivatives.

This minimization problem can be formulated in a rigorous mathematical framework given by the classical calculus of variations [14] :

We set

$$F(y, \dot{y}) = n(y(t))|\dot{y}(t)|,$$

then Fermat principle can be written as the minimization of

$$J[y] = \int_a^b F(y, \dot{y})\, dt$$

over all continuous path $y(t)$ with continuous first derivatives such that $y(a) = A$ and $y(b) = B$. a and b are fixed. The minimization problem is independent of the the time parameterization t. It must not be confused with the travel time which is the value taken by J at the optimum.

We work in the space $\mathbb{R}^d$. We implicitly assumed that $y = (y_1, y_2, ..., y_d)$, $A = (A_1, A_2, ..., A_d)$, $B = (B_1, B_2, ..., B_d)$.

The first important result of the calculus of variation (see [32] for a detective-story like investigation of this result) states that a necessary condition for y to be an extremal of the functional J is that the functions y_i satisfy the Euler equations

$$\begin{cases} F_{y_i} - \dfrac{d}{dt} F_{\dot{y}_i} = 0 \\ y_i(a) = A_i, \ y_i(b) = B_i \end{cases} \qquad (i = 1, ..., n).$$

These equations are obtained using the first variation of J.

For our purpose A is fixed and we allow B to vary. We briefly review the Hamiltonian formalism applied to this problem [14].

Let y be an extremal path. The traveltime from A to B is given by $S(B) = J[y]$. The Hamiltonian $H(y, p)$ where $p = (p_1, p_2, ..., p_d)$ is defined by

$$H(y, p) = p.\dot{y} - F(y, p),$$

$p = p(y)$ is actually a function of y, solution of the following equation

$$p_i(y) = F_{y_i}(y, \dot{y}).$$

We can rewrite the Euler equations

$$\begin{cases} \dfrac{dy_i}{dt} = H_{p_i} \\ \dfrac{dp_i}{dt} = -H_{y_i} \\ y_i(a) = A_i, \ y_i(b) = B_i \end{cases} \qquad (i = 1, ..., n).$$

$S(y(t))$ satisfies the Hamilton-Jacobi equation

$$H(y_1, y_2, ..., y_d, S_{y_1}, S_{y_2}, ..., S_{y_d}) = 0.$$

Using the particular form chosen above for F and the time parameterization $dt = n(y(t))ds$, we recover the classical ray equations

$$\begin{cases} \dfrac{dy_i}{dt} = \dfrac{p_i}{n(y)} \\ \dfrac{dp_i}{dt} = n_{y_i}(y) \\ y_i(a) = A_i,\ y_i(b) = B_i \end{cases} \qquad (i = 1, ..., n)$$

and Eikonal equation for the traveltime S

$$\sqrt{S_{y_1}^2 + S_{y_2}^2 + ... + S_{y_d}^2} = n.$$

Classical ray tracing consists in the integration of the ordinary differential ray equations. One usually replaces the two end points initial conditions by a fixed initial point $y(a) = A$ and direction $\dot{y}(a) = v$. By allowing v to vary and computing a (necessary finite) number of corresponding rays, one tries to cover as best as possible the domain of interest. The travel times are then computed along these rays. If one wants to recover the whole travel time field between these rays, an interpolation process has to be used (see [27] [29] [16] [11] [13] for example). This process is difficult and requires heuristics in zones where few rays enter (low density zones) or zones with complex multivalued travel time fields (zones where different rays cross). Figure 3 and 4 illustrates these phenomena.

An alternative to this ray tracing method is the direct resolution of the Eikonal equation in the domain of interest using upwind finite difference or finite element schemes. This approach has the advantage of giving straightforwardly a travel time field in all the domain. These schemes however only compute a single valued solution called the viscosity solution which corresponds, as explained in the next section, to the first arrival time.

The approach developed in this paper proposes to combine these two techniques so as to compute easily the multivalued travel time solution of the Eikonal equation.

For the sake of completeness, let us mention that the Eikonal equation can also be recovered as the zeroth order of an asymptotic expansion of the harmonic wave equation. Higher order terms give the amplitude equations which are also hyperbolic equations. The amplitude terms give a measure of the density of rays. On this topic see [25] [17] [10] and more generally all the literature on geometrical optics.

3 Lions's theorem on the Eikonal equation and its consequences

This section lay the theoretical bases of our method.

Lions ([18] , p. 116, 117) characterizes the viscosity solution of the Eikonal equation in terms of a semi-distance L.

Let Ω be a bounded, smooth and connected domain in $\mathbb{R}^d$. For x, y in $\overline{\Omega} \times \overline{\Omega}$, we define

$$L(x,y) = \inf_{T_0, \xi \in X(T_0)} \int_0^{T_0} n(\xi(t))dt$$

where

$$X(T_0) = \{\xi /\ \xi(0) = y_0,\ \xi(T_0) = x,\ |\dot{\xi}(t)| \leq 1 \ a.e.\ in\ [0, T_0],\ \xi(t) \in \overline{\Omega}\ \forall t \in [0, T_0]\}$$

This is an optimal control problem with state constraints on the paths. Let K denote an upper bound for the value function $S(.) = L(., y_0)$. Then, S is the unique viscosity solution of the following problem ([24] and also [?] p. 109):

$$\sqrt{S_{y_1}^2(y) + S_{y_2}^2(y) + \ldots + S_{y_d}^2(y)} = n(y)\ for\ y \in \Omega \setminus \{y_0\},\ S(y_0) = 0.$$
$$S(y_1, y_2, \ldots, y_d) > K\ for\ y \in \partial\Omega \setminus \{y_0\}$$

For all time parameterization t the curvilinear abscissa satisfy $|\dot{\xi}(t)|dt = ds$. It implies that

$$\int_x^y n(\xi(s))ds = \int_0^{T_0} n(\xi(t))|\dot{\xi}(t)|dt \leq \int_0^{T_0} n(\xi(t))dt.$$

The curvilinear abscissa is therefore a minimal time parameterization and L is formally equivalent to the travel time from x to y given by the Fermat principle of section 2.

S defined above is the viscosity solution of the Eikonal equation restricted to lie in a fixed domain Ω with a prescribed condition at point y_0. The paths ξ are shot from the point source y_0, the travel time is 0 at y_0, and are restricted to lie in $\overline{\Omega}$. It corresponds for the Hamilton Jacobi equation to the Soner boundary condition.

In a real seismic experiment the domain is unbounded and this theorem is actually also true for Ω unbounded. In this case the optimal ξ, when it exists, satisfies the ray equations and $S(\xi(t))$ is the absolute minima of $L(\xi(t), y_0)$ or, otherwise said, the first arrival time.

When Ω is bounded, only optimal path ξ lying strictly into Ω will satisfy the ray equations. This means that in some cases the infimum may be reached by a path a portion of which is on the boundary $\partial\Omega$ of Ω and therefore has no physical interpretation in terms of travel time along rays. The solution S of the Eikonal equation is now a relative minima as there may be other optimal paths giving earliest travel time which are not necessarily strictly contained in Ω.

We want to stress the importance of the following remark , illustrated in figure 1, which is the basic idea of our numerical method :

If we assume that we are in a situation where a ray gives a local minimum of J. Then, there exists a local domain Ω for which this ray is now a global minimum with respect to all the path restricted to lie in Ω. Lions 's theorem now guarantees that the viscosity solution of the Eikonal equation on Ω gives along this ray the travel time associated with it. If this local minimum is not a global minimum with respect to all possible path (i.e. not restricted to lie only in Ω) this ray gives a later travel time.

An abundant literature on the viscosity solutions of Hamilton-Jacobi equations is available, see [26] [12] for detailed references.

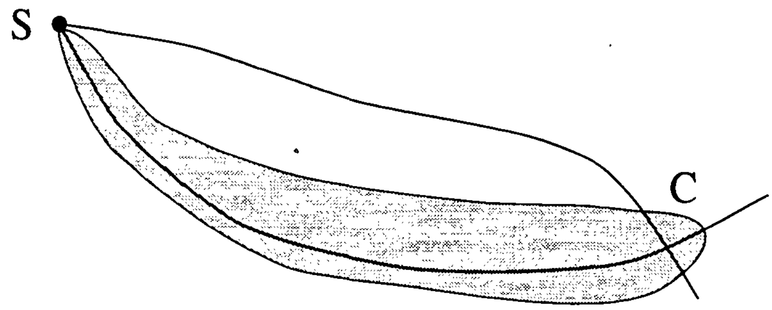

Figure 1: Two rays are crossing. One yield a first arrival time at the crossing C from the source S, the other (in the colored domain) a later arrival time. The second ray will still be a global minimum in the colored domain and therefore the viscosity solution in this domain gives the later arrival time at the crossing.

4 The sufficient conditions for a local minimum

In this section, the conditions for a ray (which is an extremum, see section 2) to be a local minimum is investigated.

The second important result of the calculus of variation (the first one is recalled in section 2) is the derivation of sufficient conditions for a ray (i.e. a solution of the Euler equation associated with the minimization of J) to be a local minimum.

In the one dimensional case (i.e. $d = 1$) for a general functional

$$J[y] = \int_a^b F(y, \dot{y})\, dt \quad y(a) = A,\ y(b) = B$$

the sufficient condition for y to be a local minimum with respect to the C^0 norm on the paths or a weak minimum in the terminology of [14] are :

i) The curve y is an extremal, i.e., satisfies Euler equation

$$F_y - \frac{d}{dt}F_{\dot{y}} = 0$$

ii) Along the curve $y = y(t)$, $F_{\dot{y}\dot{y}} > 0$.

iii) The interval $[a, b]$ contains no points conjugate to a, or equivalently, the quadratic functional

$$\int_a^b P\dot{h}^2 + Qh^2\, dt$$

where

$$P(t) > 0 \quad (a \leq t \leq b)$$

is positive definite for all $h(t)$ such that $h(a) = h(b) = 0$. Here $P = \frac{1}{2}F_{\dot{y}\dot{y}}$ and $Q = \frac{1}{2}(F_{yy} - \frac{d}{dt}F_{y\dot{y}})$ are to be evaluated along $y = y(t)$.

These conditions are derived from the second variation of J. An analog for higher dimensions ($d > 1$) exists. It is however possible (as shown in section 6) to describe complex two dimensional ray behavior in a stratified medium using the one dimensional formalism.

Let us consider the two dimensional case ($d = 2$) where the slowness n only depends on one variable, say y_2. The functional to minimize now is

$$J[y] = \int_a^b n(y_2(t))\sqrt{\dot{y_1}(t)^2 + \dot{y_2}(t)^2}\, dt$$

subject to the constraint $y_i(a) = A_i$, $y_i(b) = B_i$.

As the minimization problem is independent of the time parameterization t and $\dot{y_1}(t)$ will keep a constant sign, it is legitimate to reduce this functional to

$$J[y] = \int_{A_1}^{B_1} n(y_2(y_1))\sqrt{1 + \dot{y_2}(y_1)^2}\, dy_1$$

by the change of variable $t = y_1(t)$. We now are in the one dimensional case with

$$F(y, \dot{y}) = n(y)\sqrt{1 + \dot{y}^2}$$

We can compute more precisely conditions *i)*-*iii)* to try to determine for which velocity profiles rays are local minima.

For the particular expression of F above, we get

$$P = \frac{1}{2}\frac{n(y)}{(1+\dot{y}^2)^{\frac{3}{2}}}$$

and, using the Euler equation given in i), we obtain the following expression for Q

$$Q = \frac{1}{2}\frac{(n_{yy}(y) - \frac{n_y^2(y)}{n(y)})}{(1+\dot{y}^2)^{\frac{1}{2}}}$$

The first two conditions i) and ii) will always be satisfied for $n > 0$. Conversely, there is no evidence that condition iii) is automatically satisfied. Given a particular slowness index n one could trace a ray, compute P and Q along this ray and then try to check the positive definiteness of the quadratic functional defined in iii).

These operations are certainly non trivial. Therefore, we give instead an equivalent but more intuitive definition of a conjugate point (still found in [14]) :

Given and extremal $y = y(t)$ (i.e. a ray), the point $\tilde{M} = (\tilde{a}, y(\tilde{a}))$ is said to be conjugate to the point $M = (a, y(a))$ if $\tilde{M}$ is the limit as $y^\star$ tends to y, in the C^1 norm on paths, of the points of intersection of y and the neighboring extremals $y^\star$ drawn from the same initial point M.

From this definition, we see that conjugate points are points around which a large number of rays with similar directions concentrates (focal points for example are conjugate points).

In figures 3 4 we see a particular ray tracing. We anticipate only one conjugate point. We expect that, for many situations of interest, conjugate points will remain few and most rays will be local minima with no restriction on their length. We abusively say that a ray is a minimum when we should say that for all points B along this ray, it is a minimum for the minimization of the functional J with constraints $y(a) = A$ and $y(b) = B$.

5 A proposed algorithm for the computation of multivalued travel time fields

Based on the idea developed in the previous sections, we now propose an algorithm which, we believe, is able to compute an approximation of the multivalued traveltime field, using a finite difference resolution of the Eikonal equation.

At that point, we assume that the velocity profile is such that almost every ray is local minima for the minimization of J from the source to any point along this ray. The notion of weak minimum is here sufficient.

Section 3 demonstrates that by choosing a local domain around a given ray, this ray becomes a global minimum on this restrained domain and therefore the viscosity solution of the Eikonal equation yields on the points along this particular ray the travel time associated with it.

We now move further towards a tractable algorithm by saying that continuity arguments makes it reasonable to assume that the local domain around a ray will host other nearby rays which are also global minima in this restricted domain.

Based on these observations, we propose the following algorithm :

a)Shoot a given number of rays, say M, in regularly spaced directions. We denote these rays by $(R_i)_{i=1..M}$ and call this step the ray discretization.

b)Define around each ray R_i a local domain Ω_i also called big ray.

c)Compute the viscosity solution of the Eikonal equation on each Ω_i.

This process gives a travel time on each big ray Ω_i. Their superpositions give a multivalued travel time field.

The difficulty lies in step b. According to the simulation objective of the experiment the construction of these local domains may vary.

Our particular purpose is to compute the multivalued travel time field produced by a point source in a given domain of interest which we denote Ω.

To reach this goal, the sets Ω_i have to satisfy at least two conflicting properties. First, they have to be big enough to cover the domain Ω. Second, they have to be small enough so that they do not contain several rays which intersect. If they do, at this intersection point, the viscosity solution gives the earliest travel time given by these rays and the other is lost.

So, completing step b may be a difficult task especially as we want to define a general algorithm which does not rely on a priori knowledge on the multi-valued travel time field and is easily implemented.

For the two dimensional case, we propose the following way of completing step b):

For each ray R_i, the big ray Ω_i will be the smallest envelope containing two successively shot rays R_i and R_{i-1}.

This strategy will only give an approximation of the multivalued travel time field which precision increases with the number M of considered rays. As shown in the next section a crude ray discretization is able to capture the travel times associated with waves arriving from different directions such as direct, reflected and head waves.

6 Numerical experiments

6.1 The test case

We first describe the context of the experiment. We are in a two dimensional (1×1) square slice of ground. x is the surface axis and z the depth. The velocity index only depends on z. We are in a layered velocity case. The velocity increases with depth as indicated on figure 2.

We performed a ray tracing from the upper left corner in this medium using a third order Runge-Kutta algorithm to integrate the ray equations. We first shoot 20 rays, figure 3. And then 100 rays, figure 4.

We observe straight rays in zone of constant velocities. Rays are turning upward in zones where the gradient of the the velocity is positive. Some rays turn upward and go back to the surface, others are bent but still go through the increasing velocity layer. We observe two caustics (envelopes of rays) and

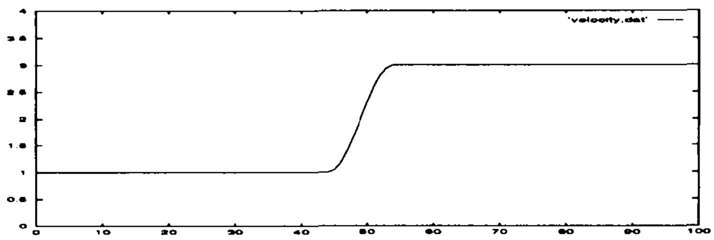

Figure 2: Velocity profile, the horizontal axis is z

where they intersect a conjugate (focal) point (see section 4). There seems to be no rays under one of these caustics but, as one can see on figure 4, increasing (possibly to infinity) the number of rays, one would succeed in filling that space. This zone has a low density of rays.

So, in spite of its apparent simplicity this test case exhibits the major difficulties of travel time computation : The travel time field is multivalued between the caustic (we observe here the classical triplication produced by the direct, reflected and head waves). In the low density zone and also for the head wave there are very few rays. Conversely near the conjugate point (focal point) a great number of rays are crowded together.

6.2 The numerical algorithm

As explained in section 5, we start by shooting M rays in regularly spaced direction (step a), in this case we chose $M = 20$ (figure 3). We then generate big rays around successive couple of rays. We simply approximate ray paths on a regular $N \times N$ (here $N = 250$) grid and define by an ad hoc algorithm the smallest possible interior zone defined by the rays (see figures 5 6 7). Special care is needed when these rays are crossing. Note that this strategy automatically generates big big rays in low density zones and small ones in zones crowded with rays.

The Soner boundary condition is enforced by imposing what is called a super-solution outside the big ray. It is done in practice by setting the boundary values to a large number. A null time is imposed at the source.

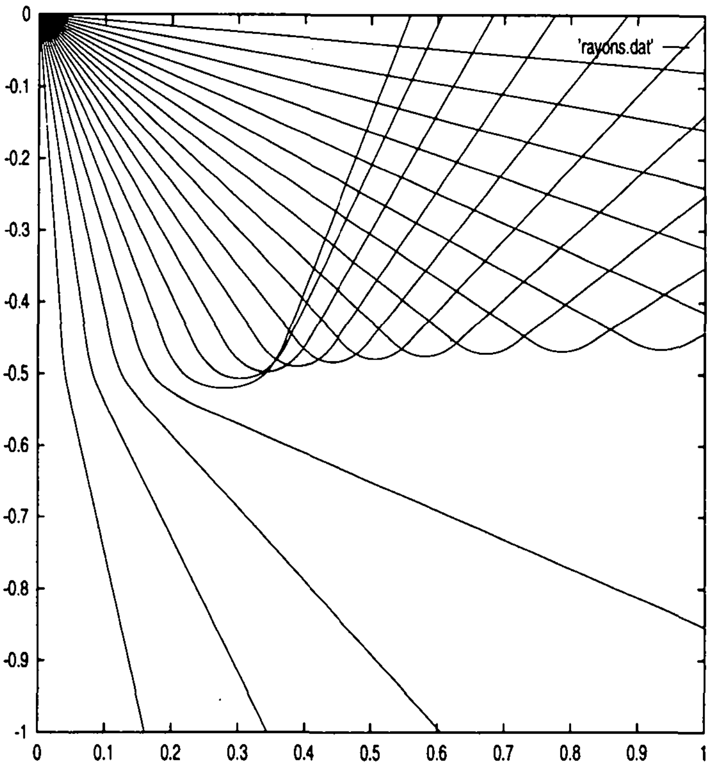

Figure 3: 20 rays shot with regularly spaced initial directions

We use an iterative scheme originally proposed by Rudin an Osher [23] to compute the viscosity solution of the Eikonal equations. This scheme was analyzed and used for the simulation of shape from shading problems in [26]. It roughly consists at each point to search for the earliest travel time computed with paths coming from its neighbors and iterate this process. This algorithm generates an increasing time field which converges to the viscosity solution. It simply mimics on a discretization grid the search for the shortest path from the source in terms of travel time. The super-solution boundary condition forces these path to remain strictly in the domain.

According to the computations done in [26] this scheme has a first order accuracy in L^1 norm and require $O(N)$ iterations to converge (this point is

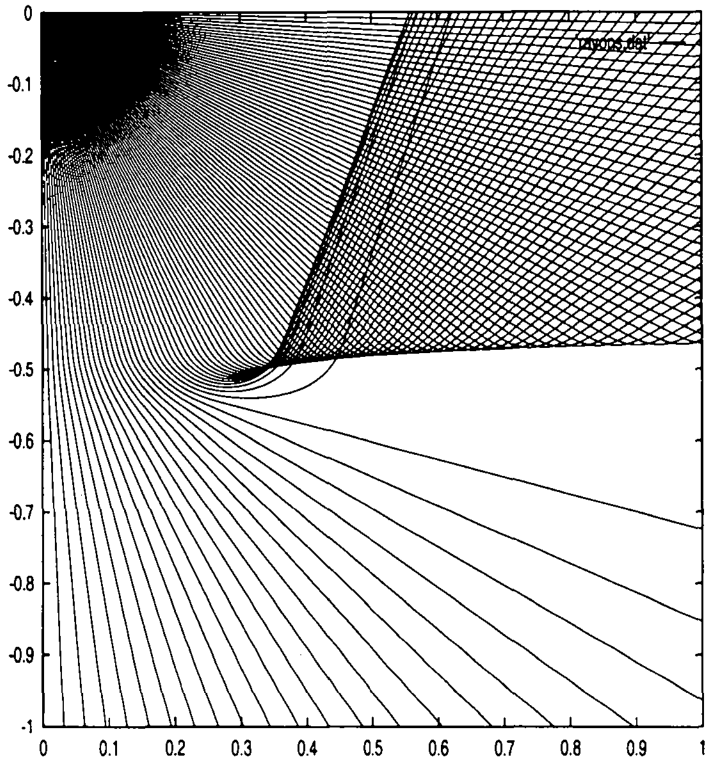

Figure 4: 100 rays shot with regularly spaced initial directions

confirmed by our experiments). This sets the computational cost of the big rays viscosity solutions to an attractive $O(M \times N)$. This cost is dominated by the cost of our (very) crude home-made algorithm generating the geometry of the big rays : $O(M \times N^2)$. It has moreover a good potential for parallelization at two levels : the core of the algorithm (the upwind scheme) which has actually been programmed on a Connection Machine and the global big rays computations which can be done separately.

The time fields in each big ray (we omitted the last two ones because of space restrictions) are displayed in figure 8 9 10. As already mentioned in the introduction we notice that the viscosity solution selects 'local' waves in each big ray.

6.3 Validation of the results as a multivalued travel time field

We compare our results to the wave fronts at different times (called snap shots) obtained from a wave equation simulation in the same medium. We recall that the solution of the Eikonal equation (geometrical optics) is also the high frequency approximation of the harmonic wave equation. It is expected to represent accurately the high frequency part of the solution of the transient wave equation. We produce snap shots of our multivalued travel time field by sorting out the contour lines of the travel times in all big rays. We allow for a small error which is proportional to the local velocity to get an accurate representation of the wave fronts.

The wave equation snap shots were graciously produced by F. Collino using his higher order absorbing boundary conditions [9]. The source is again located at the left top corner and is a $64Hz$ Gaussian derivative. This frequency is not far from the limit of current computers capabilities, we used a Cray C98 to compute the snap-shots. The amplitude scale was saturated to focus on the phase of the wave fronts, hence the visible dispersion effects.

Results are displayed in figure 11 12 13 and show a remarkable agreement. The triplication between direct reflected and head waves is perfectly computed.

On the wave equation simulation, we observe a trailing reflection phenomenon on the increasing velocity layer. Consistent with the ray tracing (figures 3 4), this reflected wave is absent from our multivalued travel time field computation. This difference is certainly produced by the low frequency part of the source term used in the wave equation simulation.

We finally notice small discrepancies on the travel time at the junctions between big rays. These errors deteriorate as the travel time increases. We input this phenomenon to the square grid approximation of the viscosity solution.

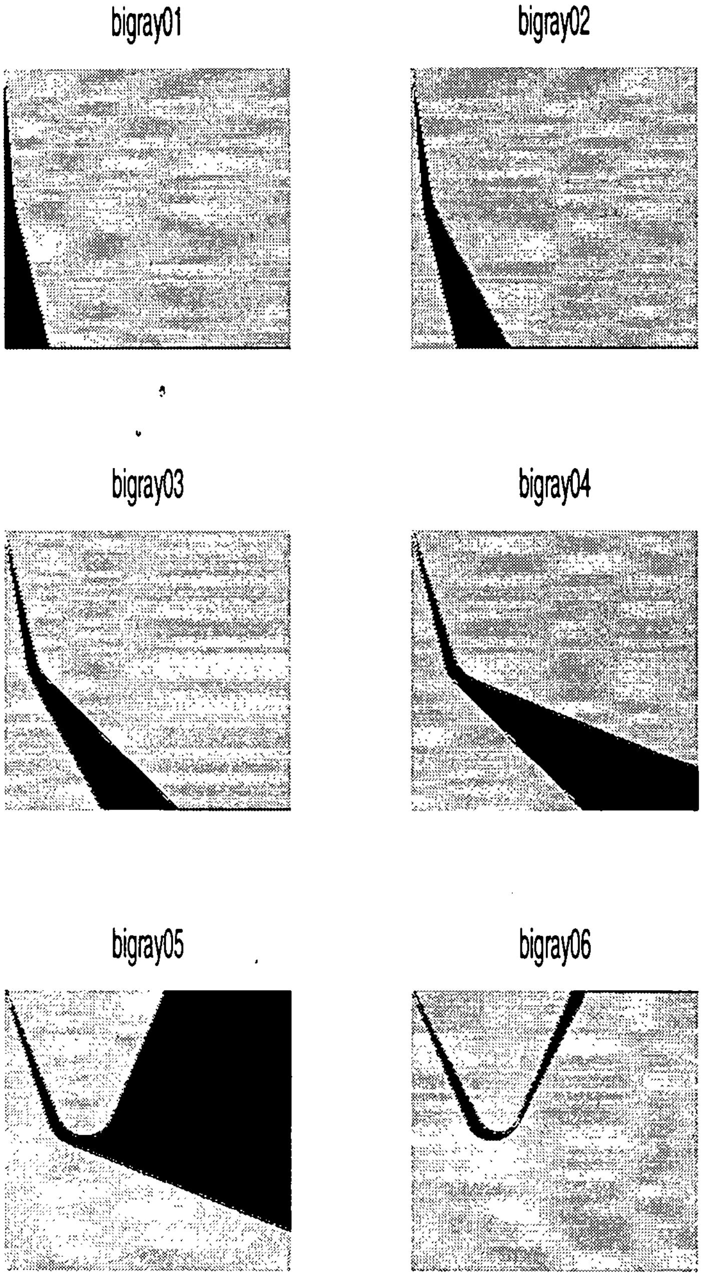

Figure 5: Big rays: in white the two rays around which the big ray envelope is generated (in black)

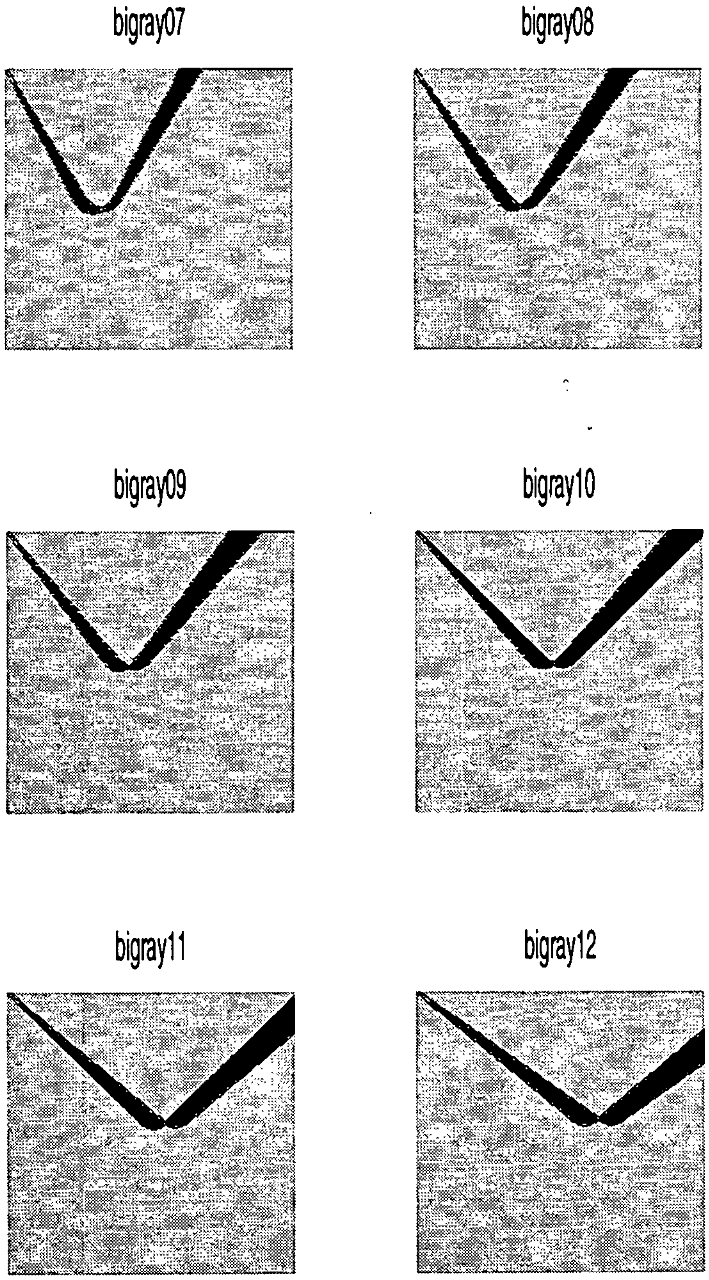

Figure 6: Big rays: in white the two rays around which the big ray envelope is generated (in black)

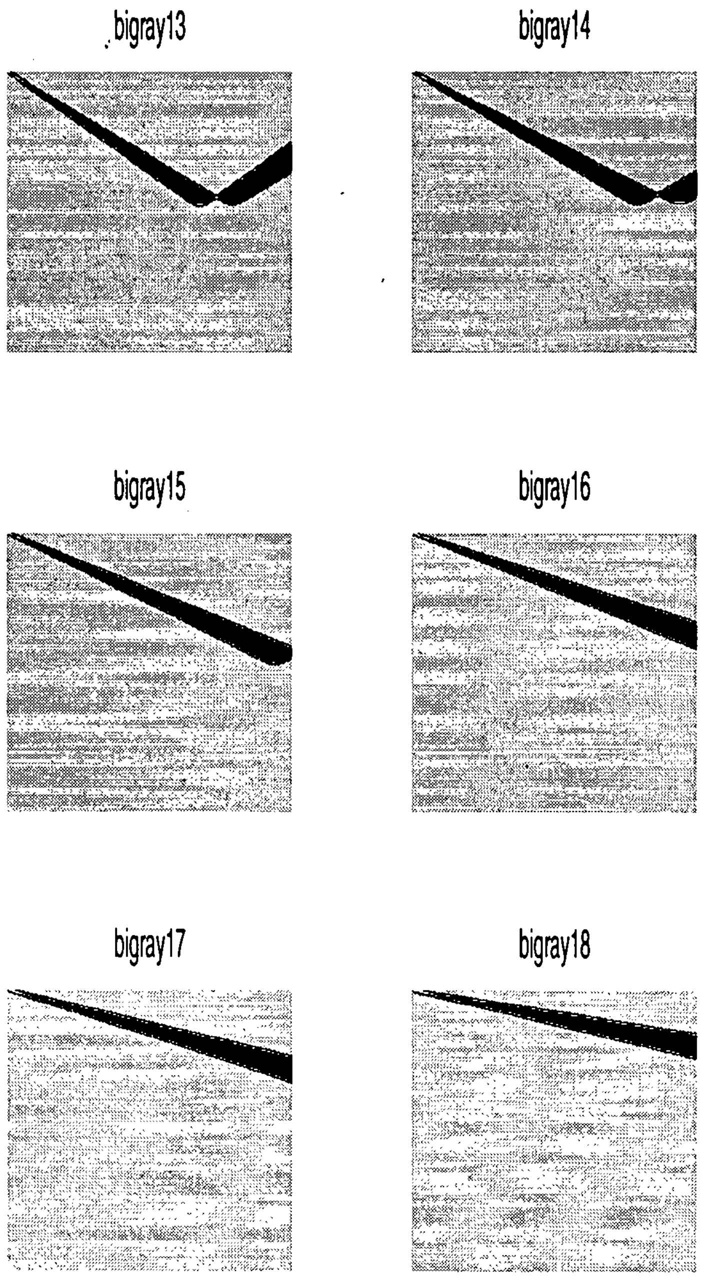

Figure 7: Big rays: in white the two rays around which the big ray envelope is generated (in black)

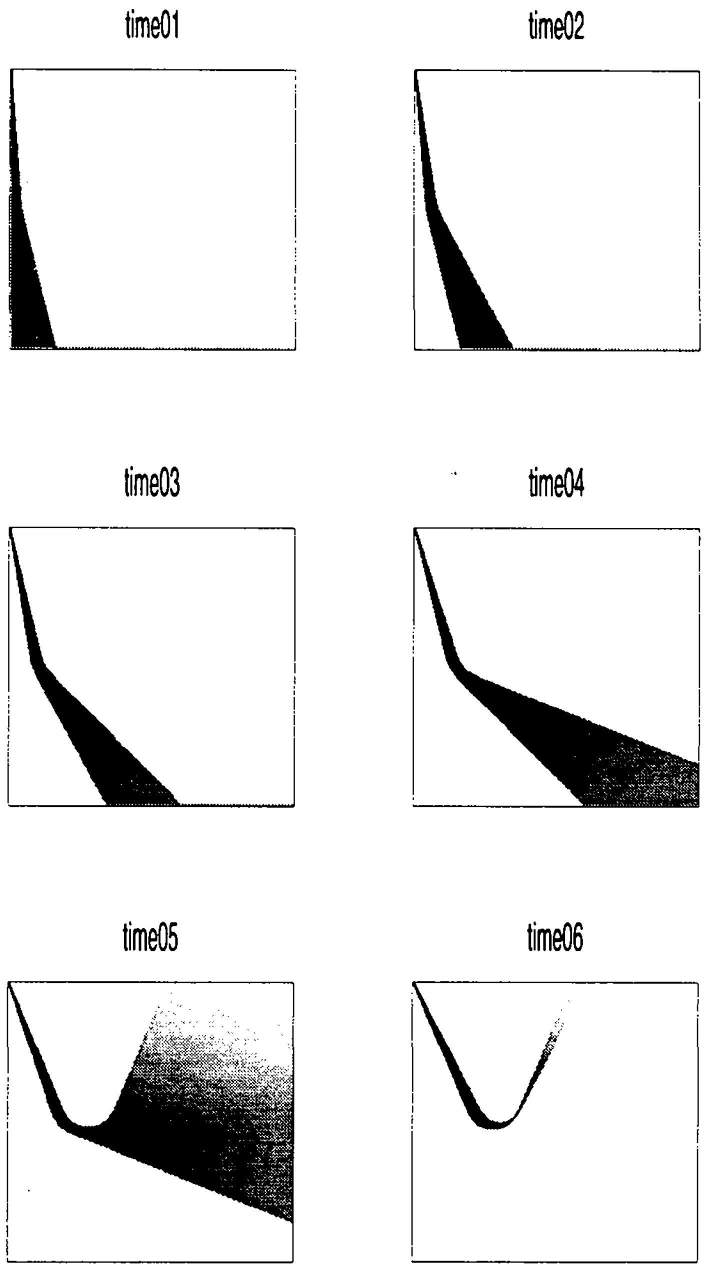

Figure 8: Travel times computed in each big ray (time increases from black to white)

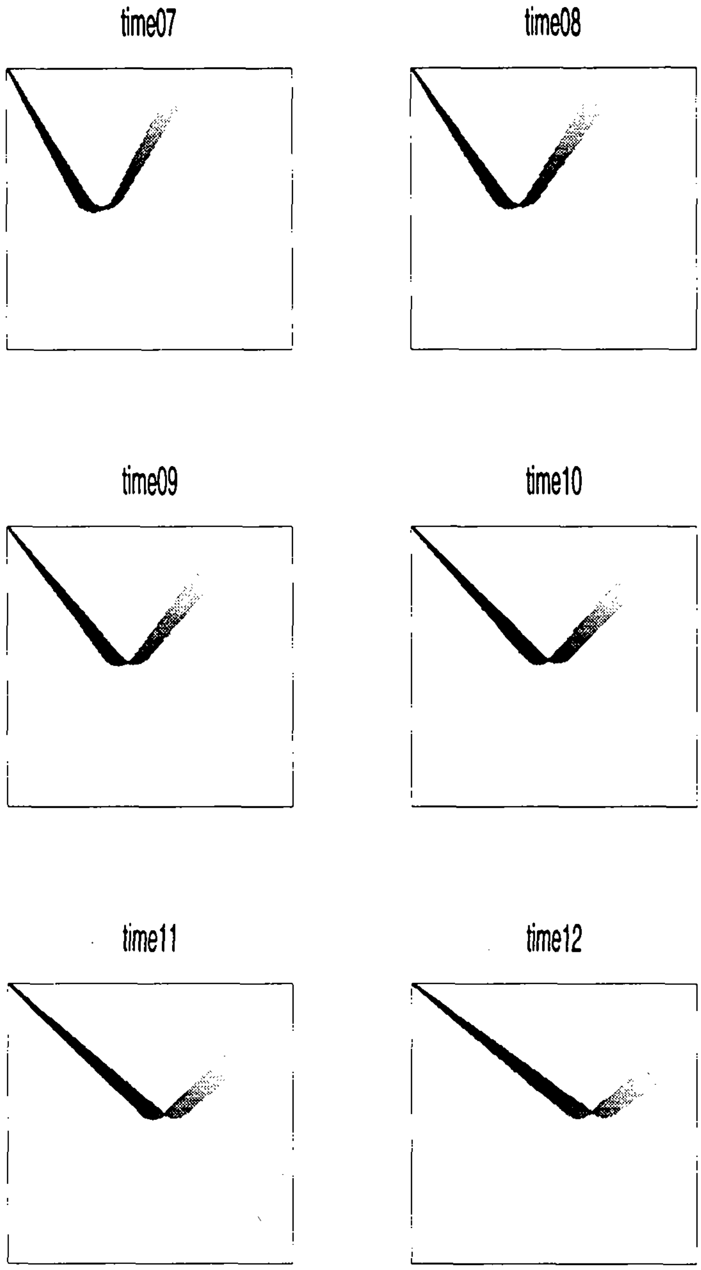

Figure 9: Travel times computed in each big ray (time increases from black to white)

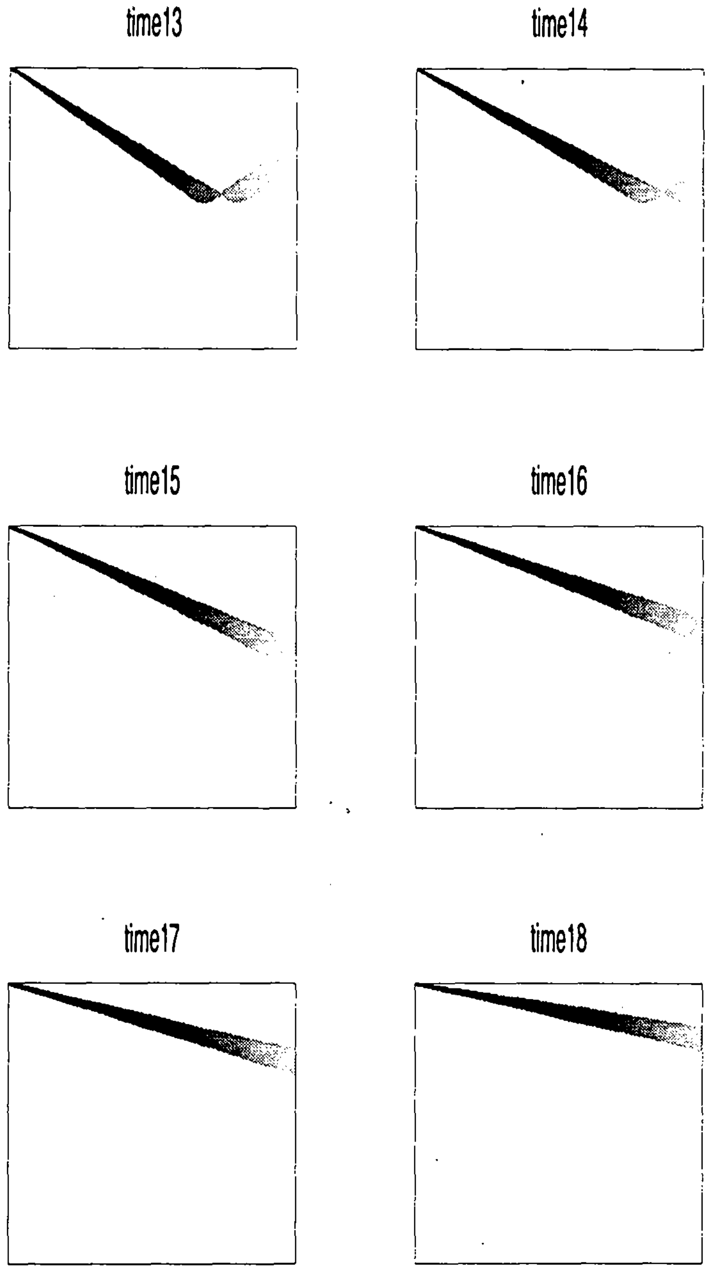

Figure 10: Travel times computed in each big ray (time increases from black to white)

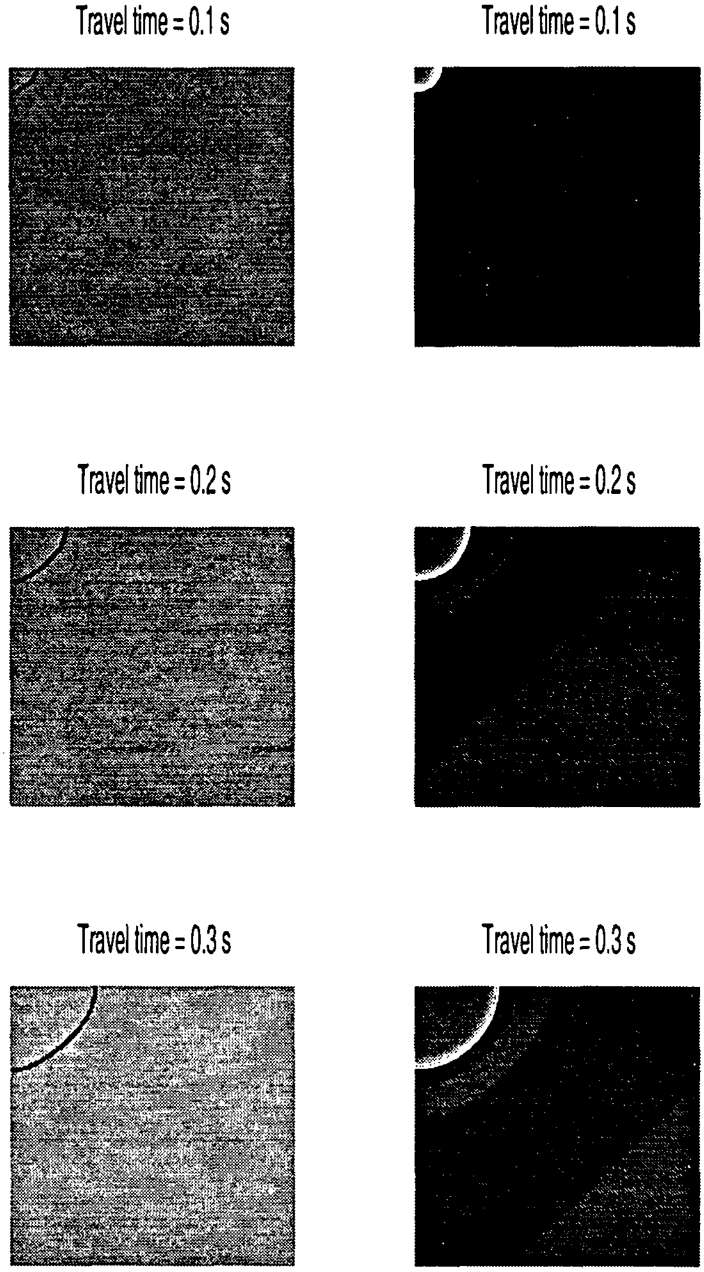

Figure 11: Left: contour lines of travel times selected from the travel times computed in the big rays. Right: snap shot of the corresponding travel time obtained from a wave equation resolution.

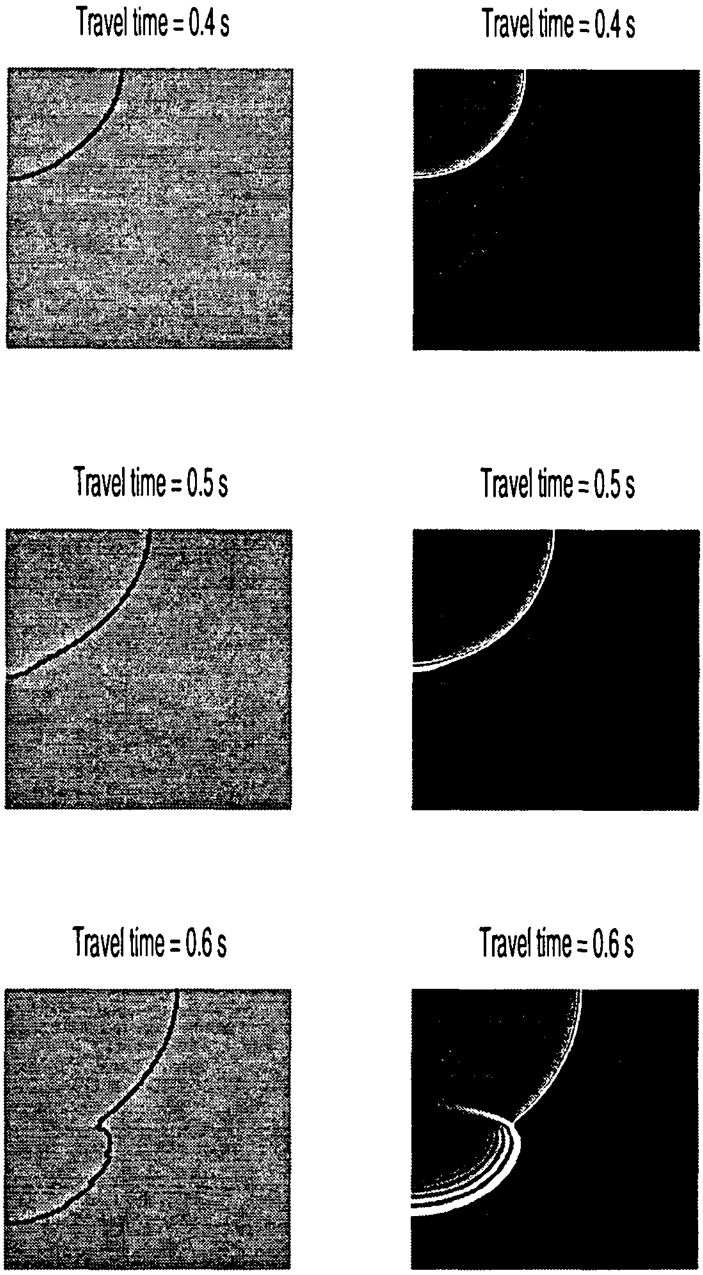

Figure 12: Left: contour lines of travel times selected from the travel times computed in the big rays. Right: snap shot of the corresponding travel time obtained from a wave equation resolution.

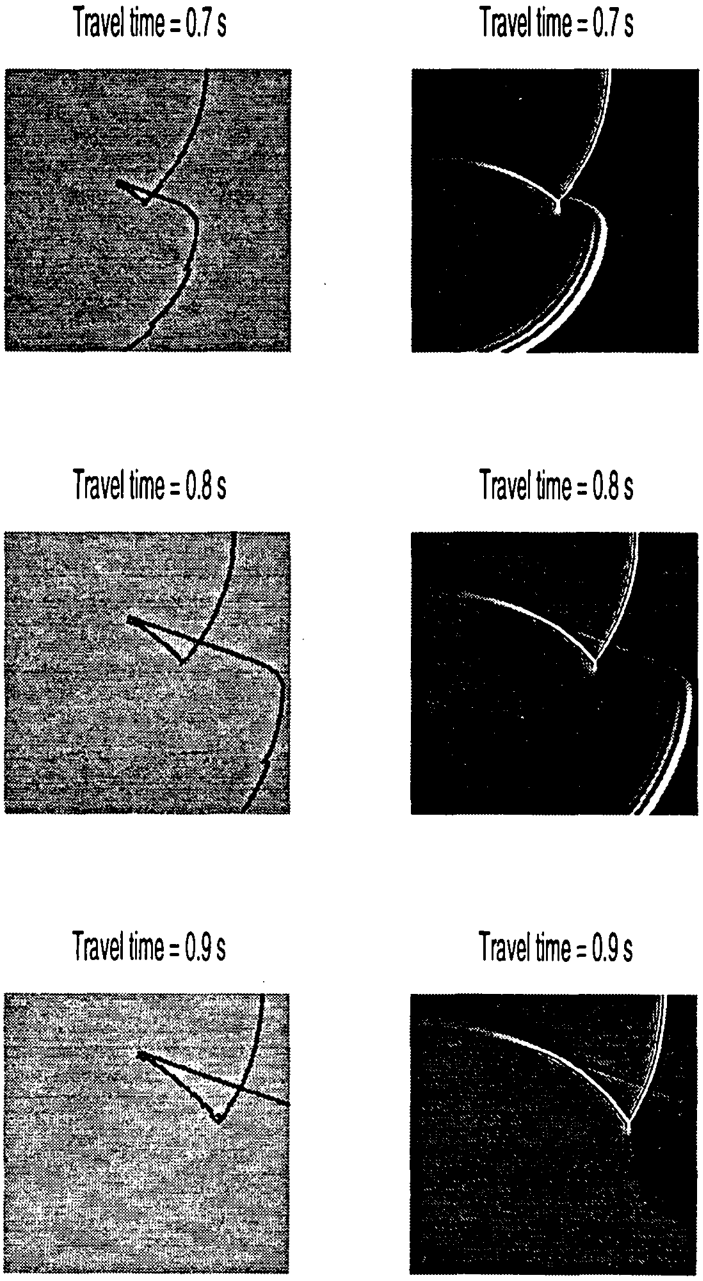

Figure 13: Left: contour lines of travel times selected from the travel times computed in the big rays. Right: snap shot of the corresponding travel time obtained from a wave equation resolution.

7 Conclusion

Using only a 20 ray discretization, we were able to recover an accurate complex multivalued travel time field using a completely automatic algorithm.

The method seems particularly attractive for low density zones where it really acts as an exact interpolator and zones with complex multivalued travel time field where the big ray superposition can be interpreted as a way to work on the support domain of the solution in phase space. By phase space here is meant the space of the positions and directions of the rays.

Except for the problem of conjugate points and possible optimal paths lying on the boundary of the big rays corresponding to non physical travel times (see section 3), the accuracy is only limited by the fineness of the finite difference grid. Because of the ray discretization we can also miss multivalued travel time produced by fine local inhomogeneities contained in one big ray. These are small scale phenomena, the scale being determined by the number of rays used in the ray discretization.

On the theoretical point of view, the question of the approximation error as a function of the ray discretization M remains open. A classification of the velocity profiles in terms of the presence of conjugate points would also be useful.

As already mentioned in section 5, one can think of other ways to generate the big rays. This problem deserves a serious study.

The most interesting numerical prospect is the 3-D extension of the method. Thanks to the super-solution boundary condition, we only need to generate the smallest envelopes containing families of neighboring rays shot in a 3-D medium to create the geometry of our big ray resolution. This is a rather non standard problem in volume meshing. We have references on an old [6] and a more promising recent approach [5] which builds a volume mesh from successive plane slices. Finally recent studies [3] have produced efficient and accurate Hamilton-Jacobi solvers on unstructured grid. It seems that all the ingredients needed for a fast and accurate 3-D multivalued travel time field solver are present.

The computation of the amplitude terms can certainly be included in our algorithm as these equations are simply hyperbolic equations using the travel time and lower order amplitude terms as coefficients. More complex ray behaviors are modelized using more sophisticated high frequency asymptotics of the harmonic wave equation [10] [19] [17] [25]. This approach leads to generalized Eikonal equations. The question of the extension of this algorithm in this case has not been investigated.

Aknowledgement

Part of this work was done while the author was a post-doctorate at the University of California at Los Angeles, Department of Mathematics.

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ISSN 0249 - 6399

RR - 2688